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NOTE

NON-REGULAR SIMPLICIAL MATROIDS

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Let $\binom{S}{k}$ denote the set of all k-element-subsets of a finite set S. A *k-simplicial matroid* on a subset E of $\binom{S}{k}$ is a binary matroid the circuits of which are simplicial complexes $\{X_1, \ldots, X_m\} \subseteq E$ with boundary 0 (mod 2). The k-simplicial matroid on $\binom{S}{k}$ is called the full simplicial matroid $G_k(S)$. The polygon matroid on the edges of a finite graph is 2-simplicial. Polygon-matroids and their duals are regular. The dual of $G_k(S)$ is $G_{n-k}(S)$ if the cardinality of S is n. More details on simplicial matroids can be found in [3, Chapter 6] and also in [4, pp. 180–181].

Welsh asked if every simplicial matroid is regular. We prove that this is not the case, for all full k-simplicial matroids $G_k(S)$ with $3 \leq k \leq n-3$ are non-regular (n is the cardinality of S). This result has also been proved by R. Cordovil and M. Las Vergnas recently. Their proof is different from our proof, which is somewhat shorter.

Let $\binom{S}{k}$ denote the set of all k-subsets of a finite set S. A *k-simplicial matroid* on a subset E of $\binom{S}{k}$ is the binary matroid on E the circuits of which are minimal simplicial complexes $\{X_1, \ldots, X_m\} \subseteq E$ with boundary 0 (mod 2). For example, the boundary of a $(k+1)$-subset (a k-simplex) is a circuit. The k-simplicial matroid on $\binom{S}{k}$ is called the full k-simplicial matroid and is denoted by $G_k(S)$. For more details on simplicial matroids we refer to the books by Crapo and Rota [3, Chapter 6] and Welsh [4, pp. 180–181].

A binary matroid is regular (or orientable) if one can partition all circuits C and cocircuits D into positive and negative parts: $C = C^+ \cup C^-$ and $D = D^+ \cup D^-$ such that

$$|C^+ \cap D^+| + |C^- \cap D^-| = |C^+ \cap D^-| + |C^- \cap D^+|.$$

Note in particular that if $C \cap D = \{x, y\}$ then x and y have the same sign in C if and only if they have different signs in D.

The orientation of a regular matroid is not unique: an element x can be moved from $+$ to $-$ in a circuit, if the sign of x is changed in all circuits and all cocircuits which contain x.

The polygon matroids of a finite graph and their duals are regular. In particular are $G_2(S)$ and $G_{n-2}(S)$ regular (n is the cardinality of S). Welsh asked if all simplicial matroids are regular [4, p. 181]. We prove that this is not the case in

Theorem. *Every full binary k-simplicial matroid $G_k(S)$ with $3 \leq k \leq n-3$ is non-regular ($n = |S|$).*

Proof. We shall first prove that $G_3(S)$ is non-regular when $n = 6$. The proof is indirect: assume that $G_3(S)$ is regular when $S = \{1, 2, \ldots, 6\}$. For brevity we shall write ijk in place of $\{i, j, k\}$. Define

$$A = \{123, 126, 134, 145, 156, 235, 245, [illegible]6, [illegible]46, 356\}$$

and

$$B = \{124, 125, 135, 136, 146, 234, 236, 2[illegible], [illegible]5, 456\}.$$

It is easy to see that A and B are circuits of $G_3(S)$; both are triangulations of the real projective plane. Note that $G_3(S)$ is selfdual: $\{X_1, \ldots, X_m\}$ is a circuit if and only if $\{S - X_1, \ldots, S - X_m\}$ is a cocircuit (cf. [3, p. 1[illegible].10]). It follows that A and B are circuits and cocircuits of $G_3(S)$.

The boundaries of 1234, 1345, 1456, 1256 and 1236 give circuits $C_1, \ldots, C_5$. The coboundaries of 13, 14, 15, 16 and 12 give cocircuits $D_1, \ldots, D_5$. Note that

$$C_i \cap D_i = C_i \cap A = D_i \cap A = \{e_i, e_{i+1}\}.$$

where $e_1 = 123$, $e_2 = 134$, $e_3 = 145$, $e_4 = 156$, $e_5 = 126$ and $e_6 = e_1$.

Choose all elements of *circuit* A positive. It follows if $G_3(S)$ is regular that e_i and e_{i+1} have different signs in the circuit C_i and same sign in the cocircuit D_i. Hence e_i and e_{i+1} have different signs in the *circuit* A for $i = 1, 2, \ldots, 5$. This gives a contradiction since $e_6 = e_1$. Therefore $G_3(S)$ can not be regular, when $n = |S| = 6$. It follows that $G_3(S)$ is non-regular when $n \geq 6$, for regularity is herédited by minors. $G_{n-3}(S)$, the dual of $G_3(S)$, is then non-regular when $n \geq 6$. Finally it follows that $G_k(S)$ is non-regular for $k \geq 3$ and $n - k \geq 3$.

The theorem has also been proved by Cordovil and Las Vergnas.

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